DEFORMED WZW MODELS AND AVERAGING IN QUANTUM GRAVITY

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We study the torus partition functions of current-current deformed WZW models in two dimensions. We decompose the WZW partition function as an orbifold sum of twisted parafermion and boson partition functions and show the deformations only change the twisted boson partition functions. Integrating over the boson moduli space defines an ensemble-averaged CFT. We compute its partition function as a Poincaré series, which suggests a 3D holographic gravity intepretation. We also calculate the density of states of the averaged theory and show that it is positive.

Dedicated to my parents, Jinrong Dong and Qing Zheng.

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CHAPTER 1

INTRODUCTION

The WZW model is one of the most famous 2D conformal field theories (CFT) and one of the most tractable CFTs. It has established connections to all parts of theoretical physics: it is the simplest quantum field theories to have a Lie group symmetry, and thus contributed greatly to chiral perturbation theory in quantum chromodynamics[29, 30]; in condensed matter physics, it is closely tied to topological phases in spin chains and the famous Haldane conjecture[1]; its connection to Chern-Simons theory also provides us with the most basic example of bulk-edge correspondence [11].

It is thus interesting to explore the space of conformal field theories using the WZW model as a starting point: after applying particular deformations to the WZW model we get a whole moduli space of CFTs. Such explorations are generally hard to investigate because deforming a CFT by an operator can violently change the algebraic structure of its operators.

In this thesis we explore moduli spaces in which the algebraic structure of the CFT is tractable. The entire moduli space can be reached by deforming the WZW model with current-current operators corresponding to symmetries in the Cartan subalgebra[12]. Not only are these deformations marginal so that they are renormalization group (RG) flow fixed points, they also preserve the operator product expansion (OPE) coefficients of the corresponding operators. These two special properties of the deformation help us make more precise statements about the deformed theories. In particular, we can analytically compute the partition functions of these theories.

In Ref. [12], it is shown that if we write the WZW model as an orbifold theory of a parafermion theory and a free boson theory, then the current-current deformations only change the partition function of the bosons, while leaving the parafermions invariant. As far as the partition function is concerned, the moduli space of the current-current deformed WZW model is locally isomorphic to the moduli space of the bosons, which is well known.

Many mathematicians have made efforts to study the moduli space of the bosons. In the mathematical context the bosonic partition function is closely related to the Siegel-Narain theta function; it is a sum over a Narain lattice. The set of all the Narain lattices is equivalent to the moduli space of the bosons. In the 1950s and the 1960s, Siegel and Weil have made numerous efforts in studying the average of Siegel-Narain theta function over the entire moduli space. Their result is called the Siegel-Weil formula [24, 25, 27, 28] and it states that the average is an Eisenstein series. We interpret the Siegel-Weil formula as a method of averaging partition functions of CFTs[2, 19, 9, 5]. Since the deformed WZW model can be written as an orbifold sum of a parafermion theory and a bosonic theory, the averaged partition function is simply the orbifold sum of a parafermion partition function and the averaged boson partition function by linearity.

In recent years, there has been evidence showing that gravitational theories might be dual to an ensemble average of theories, the most famous of which might be the 2D Jackiw-Teitelboim gravity/random matrix theory duality[8, 22, 23, 21]. In particular, gravitational wormhole geometries, a source of the factorization problem, which contribute to the gravitational path integral can be efficiently explained if the gravitational theory is dual to an ensemble of theories. This idea was particularly important in the partial resolution of the black hole information paradox; in particular, the final flattening of the Page curve results from closely related configurations[3, 20]. We wish to generalize the previous $\mathrm{AdS}_2/\mathrm{CFT}_1$ duality to higher dimensions. Two previous works[2, 19], of particular importance to this paper, used exactly the same framework to calculate the average of the free bosonic theory without the presence of parafermions; the final result of the averaged partition function is equal to the partition function of an exotic 3D $U(1)$ Chern-Simons gravity theory without instanton configurations of fields. Ref. [2] also uses the average density of states to extract the gap of such a theory, which they find to be $c/2\pi e$ in which c is the central charge. These dualities are special because unlike traditional holographic examples, there is no fundamental microscopic principle to date that explains the emergence of an ensemble averaged duality.

The central result of our computation is that the averaged partition function of the deformed $SU(N+1)_k$ WZW model can be written as a Poincaré series

$$\langle Z_{WZW}(\tau)\rangle = \sum_{\gamma\in\Gamma/\Gamma_0(\infty)} \sum_{\lambda\in\Omega_k^+} |c_0^\lambda(\gamma\tau)|^2 \tag{1.1}$$

in which $\Gamma = SL(2,\mathbb{Z})$, $\Gamma_0(\infty)$ is the subgroup generated by the T transformation, and c_μ^λ is the string function of the $SU(N+1)_k$ WZW model. The sum over modular images strongly suggests the possible interpretation of the sum above as summing over topologies in 3D gravity. Furthermore, since the central charge of our model $c = \frac{k(N^2+2N)}{k+N+1}$ is larger than N, the number of conserved currents, there is a more natural possibility to construct a dual gravitational theory. If a dual topological gravitational theory is found, its perturbative partition function must have the form

$$Z_{grav}(\tau) = \sum_{\lambda\in\Omega_k^+} |c_0^\lambda(\tau)|^2 \tag{1.2}$$

In previous work [18], a similar Poincaré series was constructed using the Virasoro vacuum character, which has the interpretation of summing over BTZ black holes in a pure 3D gravity theory. However, that Poincaré series did not make sense as a CFT, particularly because the density of states of the Poincaré series has negative values and thus violates unitarity. Various fixes, most recently adding conical defects, have been applied in order to save unitarity[4]. In our work, we explicitly check that our theory has positive density of states.

The structure of the thesis is planned as follows. In Chapter 2 we give some prerequisites on affine Lie algebra and WZW models; in Chapter 3 we study the partition function of the current-current deformed WZW models; in Chapter 4 we average over the deformations.

CHAPTER 2

$SU(N+1)$ WZW MODEL AND AFFINE LIE ALGEBRA PREREQUISITES

In this section, we will introduce the basic techniques necessary to understand the $SU(N+1)$ WZW model and its characters. In particular, we will first understand the structure of the $su(N+1)$ Lie algebra, and then the $\hat{su}(N+1)$ Lie algebra, which will finally lead to the $SU(N+1)$ WZW model. This chapter will follow Chapters 13-15 of Ref. [10] closely.

2.1 Structure of the $su(N+1)$ Lie algebra

A Lie algebra $\mathfrak{g}$ is a vector space with an antisymmetric commutator $[\ ,\ ]$ satisfying the Jacobi identity:

$$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0,\ \forall X,Y,Z\in\mathfrak{g} \tag{2.1}$$

In particular, if we consider a set of generators $\{J_a\}$ of the Lie algebra, we can define the structure constants f_c^{ab}

$$[J^a,J^b]=\sum_c i f_c^{ab} J_c \tag{2.2}$$

These structure constants characterize the Lie algebra.

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is defined as the maximal set of commuting generators $H^i\in\mathfrak{g}$ s.t.

$$[H^i,H^j]=0 \tag{2.3}$$

$\dim\mathfrak{h}=r$ is called the rank of the Lie group. The remaining generators can be organized into ladder operators that correspond to particular roots: $\alpha=(\alpha^1,\ldots,\alpha^r)$ is a root and E^α the corresponding ladder operator if

$$[H^i,E^\alpha]=\alpha^i E^\alpha \tag{2.4}$$

The lattice generated by all the roots α is called the root lattice Q. $\mathfrak{g}$ is spanned by the Cartan subalgebra and the set of ladder operators: they form a complete basis of the Lie algebra. There is a natural metric K called the Killing form on the Lie algebra:

$$K(X, Y) = \frac{1}{2g}\mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y) \tag{2.5}$$

in which $\mathrm{ad}X(Y) = [X, Y]$, and g is the dual Coxeter number of the Lie algebra. The trace is over the standard basis $\{J^a\}$ of the Lie algebra. We normalize the generators such that

$$K(H^i, H^j) = \delta^{ij} \tag{2.6}$$

and

$$K(E^\alpha, E^{-\alpha}) = \frac{2}{|\alpha|^2} \tag{2.7}$$

with all the other entries 0.

Consider a representation of the Lie algebra $\mathfrak{g}$. A basis $\{|\lambda\rangle\}$ can always be found such that

$$H^i\,|\lambda\rangle = \lambda_i\,|\lambda\rangle \tag{2.8}$$

The vector $\lambda = (\lambda_1, \ldots, \lambda_r)$ is called a weight; the weights generate the weight lattice P; P is dual to the lattice of long roots, $Q^\vee$. Since the Killing form of the Lie algebra has integer entries, roots are special weights, that is, $Q \subset P$. One can interpret the λ as living in the dual space of the root, meaning $\lambda(H^i) = \lambda_i$. Next, we will classify all the weights of $\mathfrak{g}$.

If one fixes a basis $\{\beta_i\}$ of roots, all roots can be written as

$$\alpha = \sum_i n^i \beta_i \tag{2.9}$$

α is positive if the first nonzero coordinate is positive; ν_i is called simple if it cannot be written as the sum of two positive roots. The set $\{\nu_i\}$ has cardinality r and forms a natural basis of Q. The corresponding coroots of the simple roots are defined as

$$\nu_i^\vee = \frac{2\nu_i}{|\nu_i|^2} \tag{2.10}$$

An important root called the highest root of $\mathfrak{g}$ is denoted θ defined as

$$\theta = \sum_i m^i \nu_i = \sum_i a^{i\vee} \nu_i^\vee \tag{2.11}$$

in which the sum $\sum m_i$ is maximized. $g = \sum a^{i\vee}$ is called the dual Coxeter number.

There is a bilinear form on the root lattice called the Cartan matrix:

$$A_{ij} = (\nu_i, \nu_j^\vee) \tag{2.12}$$

in which the dot product is induced by the Killing form. Although the weights can be expanded in the basis of the simple roots, the coefficients are not integers. A more convenient basis $\{\omega^i\}$ is the dual basis of $\{\nu_j^\vee\}$:

$$(\omega^i, \nu_j^\vee) = \delta_j^i \tag{2.13}$$

ω^i are called the fundamental weights. A weight λ can be expressed as

$$\lambda = \sum_i \lambda_i \omega^i \tag{2.14}$$

From now on whenever we say $\lambda = (\lambda_1, \ldots, \lambda_r)$ we mean its coordinate in the fundamental weight basis. λ_i take values in $\mathbb{Z}$. A particularly important weight called the Weyl vector ρ is defined as

$$\rho = \sum_i \omega^i \tag{2.15}$$

The dot products between the fundamental weights are defined as

$$F^{ij} = (\omega^i, \omega^j) \tag{2.16}$$

in which F is called the quadratic form matrix. The relationship between F and A is

$$F^{ij} = (A^{-1})^{ij} \frac{|\nu_j|^2}{2} \tag{2.17}$$

The Weyl group W of $\mathfrak{g}$ characterizes the symmetry for both lattices P and Q. For any element $w \in W$, w can be decomposed as

$$w = s_\alpha \dots s_\beta \tag{2.18}$$

in which s_α are called the Weyl reflections

$$s_\alpha \lambda = \lambda - (\alpha^\vee, \lambda)\alpha \tag{2.19}$$

They reflect weights in the direction of the root. $\epsilon(w) = \det(w)$ reflects the parity of the Weyl group element.

Any irreducible representation of the Lie algebra has a highest weight state $|\lambda\rangle$, such that

$$E^\alpha |\lambda\rangle = 0, \ \forall \alpha > 0 \tag{2.20}$$

All the other states in the irreducible representation can be generated by repeatedly applying combinations of ladder operators on $|\lambda\rangle$.

Since we are only concerned about the $su(N)$ Lie algebra, it is worth instantiating the abstract concepts we illustrated above.

$X \in su(N+1)$ if

$$X^\dagger = X, \mathrm{Tr}(X) = 0 \tag{2.21}$$

The Cartan subalgebra $\mathfrak{h}$ is given by the diagonal matrices in $\mathfrak{g}$. The dimension of the Cartan subalgebra is N. All simple roots are long; that is, $|\nu_i|^2 = 2$, and $\nu_i^\vee = \nu_i$. The highest root θ is $\sum_i \nu_i$. The Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \tag{2.22}$$

Sometimes the Cartan matrix is also called the Killing form because the Killing form restricted to the Cartan subalgebra $\mathfrak{h}$ in a particular basis called the Chevalley basis is exactly this Cartan matrix.

It is easy to prove using induction that $\det A = N + 1$. This determinant gives the square volume of the unit cell of the root lattice. Hence, $|P/Q| = N + 1$, an identity that will prove to be useful below.

For computational purposes, it is useful to come up with an explicit embedding of $\{\nu_i\}$ into $\mathbb{R}^{N+1}$, in the hyperplane perpendicular to the vector $(1, \dots, 1)$. The embedding is

$$\begin{aligned} \nu_1 &= (1, -1, 0, 0, \dots); \\ \nu_2 &= (0, 1, -1, 0, \dots); \\ &\dots \\ \nu_N &= (0, 0, \dots, 1, -1). \end{aligned} \tag{2.23}$$

2.2 Structure of the $\hat{su}(N+1)$ affine Lie algebra

An affine Lie algebra $\hat{g}$ is a central extension of the loop algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. with the following commutation relations:

$$[J^a \otimes t^n, J^b \otimes t^m] = \sum_c i f_c^{ab} J^c \otimes t^{n+m} + \hat{k} n K(J^a, J^b) \delta_{n+m,0} \tag{2.24}$$

in which $\hat{k}$ is a new element added into the algebra in the central extension that commutes with all the $J^a \otimes t^n$ elements. Define $J_n^a = J^a \otimes t^n$. Obviously $\{H_0^i\} \cup \{\hat{k}\}$ is an abelian subalgebra; it is not maximal because the eigenvector $(\alpha, 0)$ appears infinitely many times in the adjoint representation, in which α is a root of the original Lie algebra. We define

$$L_0 = -t\frac{d}{dt} \tag{2.25}$$

to break this degeneracy. The commutation relation is

$$[L_0, J_n^a] = -nJ_n^a \tag{2.26}$$

The resulting algebra

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}L_0 \tag{2.27}$$

is called an affine Lie algebra, with the maximal Cartan subalgebra $\{H_0^i\} \cup \{\hat{k}\} \cup \{L_0\}$.

Similarly the affine Lie algebras also have their Killing forms, roots and weights. The Killing form is defined as

$$K(J_n^a, J_m^b) = \delta^{ab}\delta_{a+b,0}, \qquad K(L_0, \hat{k}) = -1 \tag{2.28}$$

with all other entries zero. Define an affine weight $\hat{\lambda}$ to the the eigenvalues of a simultaneous eigenstate of the Cartan subalgebra:

$$\hat{\lambda} = (\hat{\lambda}(H_0^1), \ldots, \hat{\lambda}(H_0^r);\ \hat{\lambda}(\hat{k});\ \hat{\lambda}(-L_0)) = (\lambda; k_{\hat{\lambda}}; n_{\hat{\lambda}}) \tag{2.29}$$

The first r components λ corresponds to a weight in the normal Lie algebra. The extended Killing form K induces the following inner product:

$$(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_{\hat{\lambda}} n_{\hat{\mu}} + k_{\hat{\mu}} n_{\hat{\lambda}} \tag{2.30}$$

Since $\hat{k}$ commutes with all generators of $\hat{g}$ its eigenvalue on the states of the adjoint representation is zero. Hence affine roots have the form

$$\hat{\beta} = (\beta; 0; n) \tag{2.31}$$

in which β is a root in the normal Lie algebra. Define the imaginary root

$$\delta = (0; 0; 1) \tag{2.32}$$

which has zero length.

Simple roots of the affine Lie algebras can be defined as $\hat{\nu}_0 = (\theta; 0; 1)$ and $\hat{\nu}_i = (\nu_i; 0; 0)$. The associated coroots are $\hat{\nu}_0^\vee = \hat{\nu}_0$ and $\hat{\nu}_i^\vee = (\nu_i^\vee; 0; 0)$. The fundamental weights $\{\hat{\omega}^i\}$ are defined to be dual to the coroots:

$$\hat{\omega}^0 = (0; 1; 0), \hat{\omega}^i = (\omega^i; a^{i\vee}; 0) \tag{2.33}$$

which satisfy the relation $(\hat{\omega}^i, \hat{\nu}_j^\vee) = \delta^i_j$.

Affine weights can thus be expended in terms of the affine fundamental weights and δ:

$$\hat{\lambda} = \sum_i \lambda_i \hat{\omega}^i + \ell\delta \tag{2.34}$$

The level of a weight is defined to be

$$k \equiv \hat{\lambda}(\hat{k}) = (\hat{\lambda}, \delta) = \sum_i a^{i\vee} \lambda_i \tag{2.35}$$

And any weight in level k can be written as

$$\hat{\lambda} = (\lambda; k; \ell) \tag{2.36}$$

Using the two equations above we derive the important identity

$$\lambda_0 = k - (\lambda, \theta) \tag{2.37}$$

The Dynkin labels of a weight track the coordinates of an affine weight without its imaginary part. That is, if $\hat{\lambda} = \sum \lambda_i \hat{\omega}^i + \ell\delta$, then we write

$$\hat{\lambda} = [\lambda_0, \lambda_1, \ldots, \lambda_r] \tag{2.38}$$

The affine Weyl vector is defined as

$$\hat{\rho} = \sum_i \hat{\omega}^i \tag{2.39}$$

which will be important later when we calculate the characters.

The affine Weyl reflections are defined as

$$s_{\hat{\alpha}}\hat{\lambda} = \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^\vee)\hat{\alpha} \tag{2.40}$$

The relevant irreducible representations of $\hat{g}$ are called integrable highest-weight representations. There is a requirement that the highest weight $|\hat{\lambda}\rangle$ in such a representation satisfy

$$k \in \mathbb{Z}_+, \qquad k \geq (\lambda, \theta) \tag{2.41}$$

All the other states in such an irreducible representation can be produced via repeated applications of ladder operators.

Again, we use $\hat{su}(N+1)$ as an example. Since $a^{i\vee} = 1$ for $su(N+1)$, for $\hat{\lambda} = [\lambda_0; \lambda_1; \ldots \lambda_N]$, the level is

$$k = \sum_i \lambda_i \tag{2.42}$$

And hence all the irreducible representations at level 2 can be listed as an example: $[2,0,0], [0,2,0], [0,0,2], [1,1,0], [1,0,1], [0,1,1]$. We denote P_k^+ to be all the affine weights with nonnegative coordinates whose level are k. We note that $(\lambda_1, \ldots, \lambda_N)$ denotes the finite part of $\hat{\lambda}$. For fixed level k, all $\hat{\lambda} \in P_k^+$ correspond to $su(N+1)$ weights λ with nonnegative Dynkin labels whose sums are less than or equal to k. We denote this set of $su(N+1)$ weights by Ω_k^+; in short, Ω_k^+ contain the finite parts of affine roots of P_k^+.

Another important automorphism of $\hat{su}(N+1)$ is called the outer automorphism group $O(\hat{su}(N+1)) = \mathbb{Z}_{N+1}$. Its generator a acts on the Dynkin labels as

$$a[\lambda_0, \lambda_1, \ldots, \lambda_N, \lambda_{N+1}] = [\lambda_{N+1}, \lambda_0, \ldots, \lambda_{N-1}, \lambda_N] \tag{2.43}$$

It can be proven that for all $A \in O(\hat{su}(N+1))$,

$$A\hat{\lambda} = k(A-1)\hat{\omega}^0 + w_A\hat{\lambda} \tag{2.44}$$

in which w_A is a Weyl reflection. In other words, since $a^m\hat{\omega}^0 = \hat{\omega}^m$,

$$a^m\hat{\lambda} = a^m(\lambda; k; n_{\hat{\lambda}}) = (w_{a^m}\lambda + k\omega^m; k; n_{a^n\hat{\lambda}}) \tag{2.45}$$

Basically it applies a symmetry in the Weyl group to the finite part and then shifts it by k times a fundamental root. This relation will be crucial below.

2.3 $SU(N+1)_k$ WZW model and its characters

A $SU(N+1)_k$ WZW model means a WZW model with $SU(N+1)$ symmetry and at level k. It has infinitely many primary fields under the Virasoro symmetry, and generally speaking in such models it's hard to calculate the partition function. However, due to the $SU(N+1)$ symmetry, we arrange the torus partition function into a finite sum of characters over weights in P_k^+:

$$Z_{WZW}(\tau) = \sum_{\hat{\lambda} \in P_k^+} |\chi_{\hat{\lambda}}(\tau)|^2 \tag{2.46}$$

in which

$$\chi_{\hat{\lambda}}(\tau) = \mathrm{Tr}_{\hat{\lambda}} e^{2\pi i\tau(L_0 - c/24)} \tag{2.47}$$

$c = \frac{k \dim \mathfrak{g}}{k+g}$ is the central charge of the model. For $SU(N+1)_k$, $c = \frac{k(N^2+2N)}{k+N+1}$. The definition of the character is an intimidating equation: there are still infinitely many states in the trace. However, there are two ways to compute this character. As usual, we define $q = e^{2\pi i\tau}$. One is by the formula

$$\chi_{\hat{\lambda}}(q) = \frac{\sum_{w\in W} \epsilon(w)\theta_{w(\hat{\lambda}+\hat{\rho})}(q)}{\sum_{w\in W} \epsilon(w)\theta_{w\hat{\rho}}(q)} \tag{2.48}$$

in which the $\theta_{\hat{\lambda}}$ are generalized theta functions defined as

$$\theta_{\hat{\lambda}}(q) = \sum_{\alpha^\vee \in Q^\vee} q^{\frac{1}{2k}|k\alpha^\vee + \lambda|^2} = \sum_{\alpha\in Q} q^{\frac{1}{2k}|k\alpha+\lambda|^2} \tag{2.49}$$

The second equality is true because we are in $SU(N+1)$ where all roots are long.

Another more useful way that will play a crucial role in the discussion is the decomposition using string functions. Consider two weights $\hat{\lambda}$ at level k and $\hat{\mu} = (\mu; k; 0)$. Since the level is fixed, we only keep track of the finite parts λ and μ. The string function is defined as

$$c^{\hat{\lambda}}_{\hat{\mu}}(q) = q^{m_{\hat{\lambda}}(\hat{\mu})} \sum_{n=0}^{\infty} \mathrm{mult}_{\hat{\lambda}}(\hat{\mu} - n\delta) q^n \tag{2.50}$$

which has the following good properties

$$\begin{aligned} c^{\hat{\lambda}}_{w\hat{\mu}} &= c^{\hat{\lambda}}_{\hat{\mu}},\ \forall w \in W \\ c^{A\hat{\lambda}}_{A\hat{\mu}} &= c^{\hat{\lambda}}_{\hat{\mu}},\ \forall A \in O(\hat{\mathfrak{g}}) \end{aligned} \tag{2.51}$$

and

$$\begin{aligned} c^{\lambda}_{\mu} &= 0 \text{ if } \ \lambda - \mu \notin Q \\ c^{\lambda}_{\mu+k\delta} &= c^{\lambda}_{\mu} \text{ if } \ \delta \in Q \end{aligned} \tag{2.52}$$

in which we changed notation: $c^{\hat{\lambda}}_{\hat{\mu}}$ and c^{λ}_{μ} mean the same thing. $m_{\hat{\lambda}}(\hat{\mu})$ is called the relative modular anomaly defined as

$$m_{\hat{\lambda}}(\hat{\mu}) = m_{\hat{\lambda}} - \frac{|\mu|^2}{2k} \tag{2.53}$$

in which μ is the finite part of $\hat{\mu}$. The modular anomaly $m_{\hat{\lambda}}$ is defined as

$$m_{\hat{\lambda}} = \frac{|\lambda + \rho|^2}{2(k+N)} - \frac{|\lambda|^2}{2k} \tag{2.54}$$

in which λ is the finite part of $\hat{\lambda}$ and ρ is the Weyl vector defined before.

The most important property of the string functions is that it is the coefficient of the decomposition of WZW characters at level k into theta functions:

$$\chi_{\hat{\lambda}}(q) = \chi_{(\lambda;k;0)}(q) = \sum_{\mu \in P/kQ} c^{\lambda}_{\mu}(q)\theta_{\mu}(q) \tag{2.55}$$

in which

$$\theta_{\mu}(q) = \sum_{\alpha \in Q} q^{\frac{1}{2k}|k\alpha+\mu|^2} \tag{2.56}$$

can be defined similarly, but with fixed level k.

CHAPTER 3

CURRENT-CURRENT DEFORMATIONS OF THE WZW PARTITION FUNCTION

In this chapter we focus on the current-current deformation of the $SU(N+1)_k$ WZW theory. As shown in Ref. [12], it is possible to deform by current-current operators corresponding to the $SU(N+1)$ symmetry. However, since not all Lie algebra elements commute with each other, some deformations are not marginal; the only marginal deformations are the deformations by currents in an Abelian subalgebra[6]. This means that, up to an isomorphism, the currents live in the Cartan subalgebra $\mathfrak{h} = \hat{u}(1)^N$.

We will first investigate a way to decompose the theory into an orbifold of $\hat{su}(N+1)_k/\hat{u}(1)^N$ and $\hat{u}(1)^N$ theories, and then find out how the $\hat{u}(1)^N$ part of the theory gets deformed.

3.1 WZW model as an orbifold theory

3.1.1 A decomposition of the partition function

The $\hat{su}(N+1)_k/\hat{u}(1)^N$ theory is studied with great detail in Ref. [13], although with many subtle details not clarified. It describes a parafermion theory, a particular anyonic theory that contains nontrivial particle statistics. If $N = 1$, the theory reduces to a $\mathbb{Z}_k$ parafermion theory; when excitations twist around each other in this theory the wavefunction gets an additional phase of $e^{2\pi i/k}$. It is the simplest example of an abelian anyon.

The definition of the parafermion theory on the torus is

$$Z^{PF}(q) = |\eta(q)|^{2N} \sum_{\lambda\in\Omega_k^+} \sum_{\mu\in P/kP} |c_\mu^\lambda(q)|^2 \tag{3.1}$$

It has subtle differences with previous definitions of the parafermion partition functions. As different conventions can lead to significant complications in the avaraging process, we use this definition and reconcile with the usual convention in Refs. [12, 13] in the appendix.

A more general partition function that we would like to consider is the twisted partition function. Given that we are calculating our partition functions on a torus, there are different boundary conditions that one can impose on the torus. The torus we use has period lattice generated by $\{1, \tau\}$; we consider 1 as the spatial direction and τ as the temporal direction. In general, one can impose twisted boundary conditions, with the twists denoted by α and β in the τ and 1 directions respectively. We define the coordinates of α and β as

$$\alpha = \alpha^i \nu_i, \qquad \beta = \beta^i \nu_i \tag{3.2}$$

The twist measures how far we have deviated from the periodic boundary conditions, a fact that will become clear in the next section.

We define the twisted partition function of the parafermions as

$$Z_{\alpha,\beta}^{PF}(q) = |\eta(q)|^{2N} \sum_{\lambda\in\Omega_k^+} \sum_{\mu\in P/kP} e^{-\pi i\alpha\cdot(2\mu-\beta)/k+\pi i\alpha^i(B_1)_{ij}\beta^j/k} c_\mu^\lambda(q)\bar{c}_{\mu-\beta}^\lambda(\bar{q}) \tag{3.3}$$

in which B_1 is a matrix closely tied to the Killing form:

$$(B_1)_{ij} = \begin{cases} K_{ij}, \text{ if } \ i < j \\ 0, \text{ if } \ i = j \\ -K_{ij}, \text{ if } \ i > j \end{cases} \tag{3.4}$$

in which K_{ij} is the Killing form. For our Lie groups, this matrix has the following form:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.5}$$

$\alpha, \beta \in Q/kQ$. We note that it again has subtle differences compared to the version in Refs. [12, 13].

Since we have defined the twisted parafermion partition function corresponding to the $\hat{su}(N+1)/\hat{u}(1)^N$ theory, it is natural to also define the twisted partition function for the $\hat{u}(1)^N$ theory. With our own conventions, the partition function is

$$Z^b_{\alpha,\beta}(q) = |\eta(q)|^{-2N} \sum_{\mu \in P/kQ} e^{\pi i \alpha \cdot (2\mu - \beta)/k - \pi i \alpha^i (B_1)_{ij} \beta^j / k} \theta_\mu(q) \bar{\theta}_{\mu-\beta}(\bar{q}) \tag{3.6}$$

By which we have the orbifold relation

$$Z_{WZW}(q) = \frac{1}{|Q/kQ|} \sum_{\alpha,\beta \in Q/kQ} Z^{PF}_{\alpha,\beta}(q) Z^b_{\alpha,\beta}(q) \tag{3.7}$$

The proof is as follows:

$$
\begin{aligned}
\frac{1}{|Q/kQ|}\sum_{\alpha,\beta\in Q/kQ} Z^{PF}_{\alpha,\beta}(q)Z^{b}_{\alpha,\beta}(q) &= \frac{1}{|Q/kQ|}\sum_{\alpha,\beta\in Q/kQ}\sum_{\lambda\in\Omega_k^+}\sum_{\mu'\in P/kP}\sum_{\mu\in P/kQ} e^{\pi i\alpha\cdot(2\mu-2\mu')}c^{\lambda}_{\mu'}(q)\bar{c}^{\lambda}_{\mu'-\beta}(\bar{q})\theta_{\mu}(q)\bar{\theta}_{\mu-\beta}(\bar{q}) \\
&= \sum_{\beta\in Q/kQ}\sum_{\lambda\in\Omega_k^+}\sum_{\mu'\in P/kP}\sum_{\mu\in P/kQ}\delta(\mu-\mu'\in kP)c^{\lambda}_{\mu'}(q)\bar{c}^{\lambda}_{\mu'-\beta}(\bar{q})\theta_{\mu}(q)\bar{\theta}_{\mu-\beta}(\bar{q}) \\
&= \sum_{\beta\in Q/kQ}\sum_{\lambda\in\Omega_k^+}\sum_{\mu\in P/kQ} c^{\lambda}_{\mu}(q)\bar{c}^{\lambda}_{\mu-\beta}(\bar{q})\theta_{\mu}(q)\bar{\theta}_{\mu-\beta}(\bar{q}) \\
&= \sum_{\beta\in P/kQ}\sum_{\lambda\in\Omega_k^+}\sum_{\mu\in P/kQ} c^{\lambda}_{\mu}(q)\bar{c}^{\lambda}_{\mu-\beta}(\bar{q})\theta_{\mu}(q)\bar{\theta}_{\mu-\beta}(\bar{q}) \\
&= \sum_{\lambda\in\Omega_k^+}\left|\sum_{\mu\in P/kQ} c^{\lambda}_{\mu}(q)\theta_{\mu}(q)\right|^2 \\
&= \sum_{\lambda\in\Omega_k^+}|\chi_{\lambda}(q)|^2 \\
&= Z_{WZW}(q)
\end{aligned}
\tag{3.8}
$$

In which $\delta(\mu \in kP)$ means that if $\mu \in kP$ then the value is 1, and 0 otherwise. In line 4 we extended the range of the β summation because the string functions vanish if $\beta \notin Q$. In the next section, we will study the twisted $\hat{u}(1)^N$ partition function more carefully.

3.1.2 Partition function of the twisted $\hat{u}(1)^N$ theory

To find out why α and β are called twists, we derive the twisted partition function for a $\hat{u}(1)^N$ theory. The $\hat{u}(1)^N$ theory describes N compact free bosons, which makes it extremely easy to calculate.

We consider a N dimensional Euclidean lattice L, generated by elements $\{e_i\}$s. The metric for the lattice L is $e_i \cdot e_j = G_{ij}$; The dual lattice is generated by the dual basis $\{e^i\}$, which satisfy $e^i \cdot e_j = \delta^i_j$. We further consider the action

$$S = \frac{1}{2\pi}\left(G_{ij} + B_{ij}\right)\int d^2z \partial_z \Phi^i \partial_{\bar{z}} \Phi^j \tag{3.9}$$

And Φ^i is identified with $\Phi^i + 2\pi n^i$ if $n^i \in \mathbb{Z}$. G and B both have real entries; however, G is symmetric, while B is antisymmetric.

We suppose our torus is defined by vectors 1 and $\tau = \tau_1 + i\tau_2$. Canonical homology cycles A and B correspond to motions in the 1 and τ directions.

Zero modes $\Phi^i = C^i z + \bar{C}^i \bar{z}$ are marked by the following boundary conditions:

$$\int_A C^i + \int_A \bar{C}^i = 2\pi n^i, \qquad \int_B C^i + \int_B \bar{C}^i = 2\pi m^i \tag{3.10}$$

A twist by $\tilde{a}$ and $\tilde{b}$ means that one does not follow this prescription, but instead

$$\int_A C^i + \int_A \bar{C}^i = 2\pi n^i + 2\pi \tilde{b}^i, \qquad \int_B C^i + \int_B \bar{C}^i = 2\pi m^i + 2\pi \tilde{a}^i \tag{3.11}$$

in which

$$\tilde{a} = \tilde{a}^i e_i, \qquad \tilde{b} = \tilde{b}^i e_i \tag{3.12}$$

in which $\tilde{a}^i, \tilde{b}^i$ are coordinates of $\tilde{a}$ and $\tilde{b}$ in the lattice L. We can immediately solve these equations:

$$C^i = \frac{2\pi i}{2\mathrm{Im}(\tau)}[(n^i\bar{\tau} - m^i) + \tilde{b}^i\bar{\tau} - \tilde{a}^i] \tag{3.13}$$

We plug into the action

$$-S_{nm} = -\frac{\pi}{2\mathrm{Im}(\tau)}(m+\tilde{a}-\tau_1(n+\tilde{b})+i\tau_2 B\cdot(n+\tilde{b}))^2 - i\pi\tau_1(n+\tilde{b})\cdot B\cdot(n+\tilde{b}) - \frac{\pi\tau_2}{2}(B\cdot(n+\tilde{b}))^2 - \frac{\pi\tau_2}{2}(n+\tilde{b})^2 \tag{3.14}$$

in which all the dot products are done using the metric G_{ij} for the lattice coordinates and the dual metric G^{ij} for the dual coordinates.

The classical path integral is done by summing over the zero modes:

$$Z^{b,cl}_{\tilde{a},\tilde{b}} = \sum_{n,m\in L} \exp(-S_{nm}) \tag{3.15}$$

we perform Poisson summation in m:

$$\sum_{n^i,m^i\in\mathbb{Z}} \exp(-S_{nm}) \propto \sum_{m_i,n^i\in\mathbb{Z}} \exp(-2\pi\tau_2 m_i G^{ij} m_j + 2\pi i m_i(\tilde{a}^i - \tau_1(n^i+\tilde{b}^i) + i\tau_2 G^{ij}B_{jk}(n^k+\tilde{b}^k)))$$
$$\exp\left(-\frac{\pi\tau_2}{2}((n^i+\tilde{b}^i)G_{ij}(n^j+\tilde{b}^j) + (n^i+\tilde{b}^i)B_{ij}G^{jk}B_{kl}(n^l+\tilde{b}^l))\right)$$
$$\exp(-i\pi\tau_1(n^i+\tilde{b}^i)B_{ij}(n^j+\tilde{b}^j)) \tag{3.16}$$

Although the last term is explicitly one, we include it for the following purpose: once we collect the three terms, we can write the total exponential as

$$Z^{b,cl}_{\tilde{a},\tilde{b}} \propto \sum_{m_i,n^i\in\mathbb{Z}} \exp(i\pi\tau\gamma_+^2 - i\pi\bar{\tau}\gamma_-^2)\exp(2\pi i m\cdot\tilde{a}) \tag{3.17}$$

in which

$$\gamma_\pm = (m_i + \frac{B_{ij}\pm G_{ij}}{2}(n^j+\tilde{b}^j))e^i \tag{3.18}$$

This is an example of the Siegel-Narain theta function, defined in Ref. [17] as

$$\Theta_\Lambda(p,q,\tau) = \exp(\pi i p\cdot q)\sum_{\lambda\in\Lambda+q} q^{\lambda_+^2/2}\bar{q}^{\lambda_-^2/2}\exp(2\pi i\lambda\cdot p) \tag{3.19}$$

We will omit the first phase because $p\cdot q = 0$ in all the calculations below, as we will see later. The underlying lattice Λ is called the Narain lattice. The Narain lattice Λ has dimension $2N$ with signature (N,N); a vector

$$\gamma = \begin{pmatrix} \gamma_+ \\ \gamma_- \end{pmatrix} \tag{3.20}$$

belongs to Λ, in which γ_+,γ_- are both N dimensional vectors. The metric in this representation is

$$S_{\mu\nu} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{3.21}$$

A more useful basis is by defining

$$\hat{e}_j = \frac{1}{2}\begin{pmatrix} B_{ji}e^i + e_j \\ B_{ji}e^i - e_j \end{pmatrix}, \qquad \hat{e}^j = \begin{pmatrix} e^j \\ e^j \end{pmatrix} \tag{3.22}$$

In which $\hat{e}_j \cdot \hat{e}^i = \delta^i_j$, and all other dot products vanish. Hence, for the vector in Eq. (3.18),

$$\gamma = n^i \hat{e}_i + w_i \hat{e}^i + q = n^i \hat{e}_i + w_i \hat{e}^i + \tilde{b}^i \hat{e}_i \tag{3.23}$$

in which $\tilde{b}^i$ is the coordinates of $\tilde{b}$ in the lattice L. Physically speaking, n is the momentum and w is the winding of the states. We notice that in this basis, the lattice vectors have integer coordinates. However, they explicitly depend on the choice of lattice L and the magnetic field B; more importantly, noting that the theta function is invariant under rotations of the lattice L, the most important components are the metric G and the magnetic field B. They are called the moduli of the Narain lattice Λ.

Furthermore, if we take

$$p = \tilde{a}^i \hat{e}_i \tag{3.24}$$

in which $\tilde{a}^i$ are the coordinates of $\tilde{a}$ in the lattice L, then it's easy to check that

$$Z^{b,cl}_{\tilde{a},\tilde{b}} \propto \Theta_\Lambda(\tilde{a}^i \hat{e}_i, \tilde{b}^i \hat{e}_i, \tau) \tag{3.25}$$

We warn the reader that p, q are $2N$ dimensional vectors whose coordinates refer to the Narain lattice basis, while $\tilde{a}$ and $\tilde{b}$ are N dimensional whose coordinates refer to the basis of L. Since $\hat{e}_i \cdot \hat{e}_j = 0$, $p \cdot q = 0$. The calculation of the quantum partition function is standard and the result is

$$Z^b_{\tilde{a},\tilde{b}} = |\eta(q)|^{-2N} \Theta_\Lambda(\tilde{a}^i \hat{e}_i, \tilde{b}^i \hat{e}_i, \tau) \tag{3.26}$$

We see that to establish a connection with the result before, we need to choose a particular lattice Λ_k and twists $\tilde{a}, \tilde{b}$ such that the corresponding twisted Siegel-Narain Theta function satisfies

$$\Theta_{\Lambda_k}(\tilde{a}^i \hat{e}_i, \tilde{b}^i \hat{e}_i, \tau) = \sum_{\mu \in P/kQ} e^{\pi i \alpha \cdot (2\mu - \beta)/k - \pi i \alpha^i (B_1)_{ij} \beta^j / k} \theta_\mu(q) \bar{\theta}_{\mu-\beta}(\bar{q}) \tag{3.27}$$

First of all, we come up with choices of G and B. This choice has been developed by Ref. [14] and reviewed in Ref. [16]. The choice is by taking

$$(G_k)_{ij} = kK_{ij} \tag{3.28}$$

in which K_{ij} is the Killing form induced inner product of the roots, and

$$(B_k)_{ij} = \begin{cases} (G_k)_{ij}, \text{ if } i < j \\ 0, \text{ if } i = j \\ -(G_k)_{ij}, \text{ if } i > j \end{cases} \tag{3.29}$$

Note that $B_1 = B_k/k$. We emphasize that since $G_k = kK$, the lattice L is *not* the root lattice Q; it is $\sqrt{k}Q$; the dual lattice $L^* = P/\sqrt{k}$. And thus, the coordinates of $\tilde{a}$ and $\tilde{b}$ are *not* the ones we naively expect from the respective coordinates of α and β.

It is crucial to observe that $((B_k)_{ij} \pm (G_k)_{ij})/2$ have k times integer entries. That is, whenever we apply $(B_k \pm G_k)/2$ to a root vector, it becomes k times a weight. In particular, γ_L, γ_R in Eq. (3.18) are both weights divided by $\sqrt{k}$. Thus, the lattice Λ_k corresponds to

$$\Lambda_k = \{(x, x') \in P/\sqrt{k} \times P/\sqrt{k} | x - x' \in \sqrt{k}Q\} = \frac{1}{\sqrt{k}}\{(x, x') \in P \times P | x - x' \in kQ\} \tag{3.30}$$

We choose

$$\tilde{a}^i = \frac{\alpha^i}{k}, \tilde{b}^i = \frac{\beta^i}{k} \tag{3.31}$$

And thus by the definition in Eq. (3.12),

$$\tilde{a} = \tilde{a}^i e_i = \frac{\alpha^i}{k}\left(\sqrt{k}\nu_i\right) = \frac{\alpha}{\sqrt{k}}, \quad \tilde{b} = \tilde{b}^i e_i = \frac{\beta^i}{k}\left(\sqrt{k}\nu_i\right) = \frac{\beta}{\sqrt{k}} \tag{3.32}$$

Thus

$$\begin{aligned}
\Theta_{\Lambda_k}(\frac{\alpha^i\hat{e}_i}{k},\frac{\beta^i\hat{e}_i}{k},\tau) &= \sum_{\lambda\in\Lambda+\beta^i\hat{e}_i/k} q^{\lambda_+^2/2}\bar{q}^{\lambda_-^2/2}\exp(2\pi i\lambda\cdot\frac{\alpha^i\hat{e}_i}{k}) \\
&= \sum_{n\in\sqrt{k}Q,m\in P/\sqrt{k}} \exp(i\pi\tau\gamma_+^2 - i\pi\bar{\tau}\gamma_-^2)\exp(2\pi i m\cdot\tilde{a}) \\
&= \sum_{m\in P/\sqrt{k},n\in\sqrt{k}Q} q^{m^2/2}\bar{q}^{(m-n-\tilde{b})^2/2}\exp(2\pi i(m_i-(B_k+G_k)_{ij}(n^j+\tilde{b}^j)/2)\cdot\tilde{a}^i) \\
&= \sum_{m\in P,n\in Q} q^{m^2/2k}\bar{q}^{(m-kn-\beta)^2/2k}\exp(2\pi i(m_i-(B_1+G_1)_{ij}(kn^j+\beta^j)/2)\cdot\alpha^i/k) \\
&= \sum_{m\in P/kQ,n,n'\in Q} q^{(m+kn)^2/2k}\bar{q}^{(m+kn'-\beta)^2/2k}\exp(2\pi i(m-\beta/2)\cdot\alpha/k-\pi i\alpha^i(B_1)_{ij}\beta^j/k) \\
&= \sum_{\mu\in P/kQ} e^{\pi i\alpha\cdot(2\mu-\beta)/k-\pi i\alpha^i(B_1)_{ij}\beta^j/k}\theta_\mu(q)\bar{\theta}_{\mu-\beta}(\bar{q})
\end{aligned} \tag{3.33}$$

3.2 Deformations and moduli space

It is shown in Ref. [12] that current-current deformations of the WZW model only concerns the $\hat{u}(1)^N$ part of the theory. In particular, it deforms the lattice Λ_k and the twists $\tilde{a}^i\hat{e}_i, \tilde{b}^i\hat{e}_i$ by a matrix O from $O(N,N)$:

$$Z^{deform,cl} \propto \Theta_{O\Lambda}(O\tilde{a}^i\hat{e}_i, O\tilde{b}^i\hat{e}_i,\tau) \tag{3.34}$$

However, $O\tilde{a}^i\hat{e}_i = (\alpha^i/k)(O\hat{e}_i)$. We notice that the coordinates α^i/k do not change at all after the deformation.

Here we must consider seriously which O matrices can quantitatively change the theta function. First of all we want to know which matrices O fix the theta function.

Apparently, if $O\Lambda = \Lambda$ then it does not change the theta function; this group of matrices live in $O(N,N,\mathbb{Z})$[12]. Another option is that O does change the lattice, but holds $|\lambda_\pm|^2$ fixed; this group of elements live in $O(N) \times O(N)$.

In short, the space of matrices O[1] that quantitatively change the Siegel-Narain theta function live in the double coset space, which is the moduli space we consider in this thesis:

$$M_{WZW} = O(N) \times O(N) \backslash O(N,N)/O(N,N,\mathbb{Z}) \tag{3.35}$$

Furthermore, it can be shown that the effect of O on Λ is simply changing G and B. It is thus worth defining a quantity involving the coordinates of the twist elements and the moduli of the lattice:

$$f_{\Lambda,k}(r^i, s^i, \tau) = \sum_{w_i, n^i \in \mathbb{Z}} \exp\Big(-\frac{2\pi i \tau_1}{k} w_i(kn+s)^i + \frac{2\pi i}{k} w_i r^i - 2\pi\tau_2 (G^{ij} v_i v_k + G_{ij}(\frac{kn+s}{2k})^i (\frac{kn+s}{2k})^j)\Big) \tag{3.36}$$

where

$$v_i = w_i - \frac{B_{ij}(kn+s)^j}{2k} \tag{3.37}$$

And thus

$$\Theta_{O\Lambda}(\frac{\alpha^i \hat{e}_i}{k}, \frac{\beta^i \hat{e}_i}{k}, \tau) = f_{O\Lambda,k}(\alpha^i, \beta^i, \tau) \tag{3.38}$$

An important quantity is the metric of the Narain lattice. For $O\Lambda$ the (negative) inner product in the (w_i, n^i) representation is

$$2S_0 = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{3.39}$$

Note the factor of 2!

We notice that Θ to f correspondence is not unique. In fact, it is true up to a scaling:

$$f_{O\Lambda,k}(\alpha^i, \beta^i, \tau) = \Theta_{O\Lambda}(\frac{\alpha^i \hat{e}_i}{k}, \frac{\beta^i \hat{e}_i}{k}, \tau) = f_{O\Lambda,1}\left(\frac{\alpha^i}{k}, \frac{\beta^i}{k}, \tau\right) \tag{3.40}$$

[1]We note that the true moduli space of the deformed WZW model is actually infinitely larger because not all matrices in $O(N,N,\mathbb{Z})$ fix the OPE coefficients. This will not be a problem here because we are only concerned about averaging the partition function; taking our smaller moduli space is the same as taking a regulator for the larger moduli space.

A mathematician-friendly notation for the functions $f_{O\Lambda,1}(0,s^i,\tau)$ is

$$f_{O\Lambda,1}(0,s^i,\tau) = f_{H,S}(s,\tau) = \sum_{x\in\mathbb{Z}^{2N}} \exp\Big(2\pi i\tau_1 S[x+s] - 2\pi\tau_2 H[x+s]\Big) \tag{3.41}$$

in which S is half the negative metric

$$S = -\frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{3.42}$$

and

$$H = \begin{pmatrix} G^{-1} & \frac{1}{2}G^{-1}B \\ -\frac{1}{2}BG^{-1} & \frac{1}{4}(G - BG^{-1}B) \end{pmatrix} \tag{3.43}$$

in which

$$S[x+s] = (x+s)\cdot S\cdot(x+s), \qquad H[x+s] = (x+s)\cdot H\cdot(x+s) \tag{3.44}$$

The representation we are using is

$$x = (w_i, n^i),\, s = (0, s^i) \tag{3.45}$$

In the new notation, there are further rescaling relations:

$$f_{H,S}(s,\tau) = f_{aH,aS}\left(s, \frac{\tau}{a}\right) \tag{3.46}$$

In the original notation of the lattice moduli G and B, it corresponds to fixing the moduli, and considering a larger lattice $\hat{e}'_i \to \sqrt{a}\hat{e}_i$, $\hat{e}^{*i'} = \sqrt{a}\hat{e}^{*i}$, whilst keeping all coordinates n^i, w_i, s^i fixed.

The two subscipts H and S must satisfy the following relation:

$$H\cdot S^{-1}\cdot H = S \tag{3.47}$$

Notice that S is fixed up to a scaling throughout, but H is explicitly dependent on G and B. Hence, actually there are three equivalent ways of describing the moduli space we are considering, which we conjecture to be a finite volume submoduli space of the total moduli space of the deformed WZW model.

1. The double coset space $O(N) \times O(N) \backslash O(N.N)/O(N, N, \mathbb{Z})$;
2. The whole space of G and B;
3. The whole space of positive definite matrices H s.t. $HS^{-1}H = S$.

The second description is most useful for us if we care about a specific point in the moduli space; the third description is most useful when we do the averaging procedure.

CHAPTER 4

THE AVERAGING PROCESS

In this section, we will carry out the averaging process for the twisted partition functions $f_{O\Lambda,k}(\alpha^i,\beta^i,\tau)$. In particular, the averaging process is known to the mathematicians as the Siegel-Weil formula. This formula describes the relationship of averaged Siegel-Narain theta functions and an Eisenstein series. In this chapter, we will investigate multiple ways to extend the Siegel-Weil formula first found in Refs. [24, 25, 27, 28] and described in detail in Ref. [26], and their application to the average of the deformed WZW partition function.

4.1 The Averaging of partition functions $Z^{b,cl}_{0,\beta}$

We recall that $Z^b_{0,\beta} = |\eta(q)|^{-2N} f_{H,S}(\frac{b}{k},\tau) = |\eta(q)|^{-2N} f_{kH,kS}(\frac{b}{k},\frac{\tau}{k})$, in which the Narain twist $p = 0$, $q = b/k = (0,\beta^i)/k$. We would like to average over the entire set of positive definite H satisfying the relation $HS^{-1}H = S$. It is worth noting that, the Siegel-Weil formula is only applicable when $2Sq$ has integer entries; since q has entries of integers divided by k, we use our rescaling technique introduced in the last section to scale S and H by k times so that this criteria is satisfied. One can in general prove that scaling by ℓk times gives the same result for arbitrary $\ell \in \mathbb{Z}^+$.

Before actually introducing the Siegel-Weil formalism, we note that there are shortcuts via the method of integration by parts one can use to do the averaging, which is described in the appendix. That method is especially powerful in the case of $b \neq 0$.

The Siegel-Weil formalism tells us that

$$\bar{f}(0,\frac{b}{k},\tau) = \int dH f_{kH,kS}(\frac{b}{k},\frac{\tau}{k})$$
$$= \gamma_b + \frac{1}{k^N}\sum_{(c,d)=1,d>0} d^{-2N}(\frac{\tau}{k}-\frac{c}{d})^{-N/2}(\frac{\bar{\tau}}{k}-\frac{c}{d})^{-N/2}\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}S[g+\frac{b}{k}]) \tag{4.1}$$

in which g is a $2N$ dimensional $\mathbb{Z}/d\mathbb{Z}$ valued vector and $\gamma_b = 0$ unless $b = 0$, in which case $\gamma_b = 1$. We note that pairs (c, d) where $d > 0$ can be mapped to elements in $SL(2,\mathbb{Z})$. That is, we write matrices in $SL(2,\mathbb{Z})$ as

$$\gamma = \begin{pmatrix} f & g \\ c & -d \end{pmatrix} \tag{4.2}$$

We simplify the last sum of phases as follows. First we write $g = (m, n)$ in which m, n are N dimensional integer mod d vectors:

$$\begin{aligned}
&\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}kS[g+\frac{b}{k}])) \\
=&\sum_{m,n \bmod d}\exp(2\pi i\frac{c}{d}km\cdot(n+\frac{b}{k})) \\
=&\prod_{i=1}^{N}\sum_{m_i,n^i=1}^{d}\exp(2\pi i\frac{c}{d}km_i(n^i+\frac{b^i}{k})) \\
=&\prod_{i=1}^{N}\sum_{n_i=1}^{d}\frac{1-\exp(2\pi icb^i)}{1-\exp(2\pi i\frac{c}{d}k(n^i+\frac{b^i}{k}))}
\end{aligned} \tag{4.3}$$

We note that this expression is zero if the denominator is not zero because the numerator is always zero. When the denominator is zero, the fraction is d by l'Hospital's rule.

And thus

$$\begin{aligned}
&\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}kS[g+\frac{b}{k}])) \\
=&d^N\prod_{i=1}^{N}\#\text{ sols of } kn^i+b^i \bmod d = 0
\end{aligned} \tag{4.4}$$

where $n^i \in \mathbb{Z}/d\mathbb{Z}$.

And now we have a lemma:

Lemma 1. *The number of solutions of*

$$kx + b \; mod \; d = 0 \tag{4.5}$$

is equal to (k, d) *when* $(k, d)|b$*; otherwise it is zero.*

Proof. When (k, d) does not divide b, the equation obviously has no solutions; any solution would introduce a paradox because one side is divisible by (k, d) and the other side is not.

Otherwise consider first the case $(k, d) = 1$. The proposition is obvious because k is multiplicative invertible mod d.

Suppose $a = (k, d) \neq 1$. Define $k' = k/a, d' = d/a, b' = b/a$. Then the equation

$$k'x + b' \bmod d' = 0 \tag{4.6}$$

has exactly one solution; call that x_0. This means $k'x_0 + b' = cd'$, or $kx_0 + b = cd$.

Consider any solution of the original equation y; that means $ky + b = d$. This means that $k'(y - x_0) = (d - c)d'$ This means that $y - x_0$ is divisible by d'. Thus, $y = x_0 + pd'$; since $y \in \mathbb{Z}/d\mathbb{Z}$, p can take values from 1 to a. This means that there are $a = (d, k)$ solutions.

$\square$

In the end, we are able to collect all the terms:

$$\begin{aligned}
\bar{f}(0, \frac{b}{k}, \tau) &= \gamma_b + \frac{1}{k^N} \sum_{(c,d)=1, d>0} d^{-N} (\frac{\tau}{k} - \frac{c}{d})^{-N/2} (\frac{\bar{\tau}}{k} - \frac{c}{d})^{-N/2} \prod_{i=1}^{N} \# \text{ sols of } kn^i + b^i \bmod d = 0 \\
&= \gamma_b + \frac{1}{k^N} \sum_{(c,d)=1, d>0} d^{-N} (\frac{\tau}{k} - \frac{c}{d})^{-N/2} (\frac{\bar{\tau}}{k} - \frac{c}{d})^{-N/2} (k, d)^N \delta(b \bmod (k, d)) \\
&= \gamma_b + \frac{1}{k^N} \sum_{(c,d)=1, d>0} d^{-N} (\frac{\tau}{k} - \frac{c}{d})^{-N/2} (\frac{\bar{\tau}}{k} - \frac{c}{d})^{-N/2} (k, d)^N \delta(cb \bmod (k, d))
\end{aligned} \tag{4.7}$$

The last line is true because b has entries less than or equal to k, and $(c, d) = 1$. Now we write $k = (k, d)p, d = (k, d)q, (p, q) = 1$. Furthermore we note that $(pc, q) = 1$. Then

$$\begin{aligned} \bar{f}(0, \frac{b}{k}, \tau) &= \gamma_b + \sum_{(pc,q)=1, q>0} |q\tau - pc|^{-N} \delta(pcb \bmod k) \\ &= \gamma_b + \sum_{(c,d)=1, c>0} |c\tau - d|^{-N} \delta(db = 0 \bmod k) \end{aligned} \tag{4.8}$$

in which in the last equality we have renamed all the variables. It is worth mentioning that the renaming of the variables do not cause any multiplicity troubles. That is, there is a bijection between pairs (c, d) and pairs (pc, q): the map from left to right is the definition. The map from the right to left is by observing that $(k, pc) = p(k/p, c) = p((k, d), c) = p$, so $(k, p) = k/(k, pc)$ and thus $d = k/(k, pc)q$, $c = pc/(k, pc)$.

This completes the averaging process for $Z^{b,cl}_{0,\beta}$.

4.2 The Averaging of partition functions $Z^{b,cl}_{\alpha,\beta}$

We note the following identity of Theta functions:

$$f_{\Lambda,1}(\frac{\alpha}{k}, \frac{\beta}{k}, \tau) = \sum_{h \bmod k} e^{-4\pi i h \cdot S \cdot a/k} f_{H,S}(\frac{h}{k} + \frac{b}{k^2}, k^2\tau) = \sum_{h \bmod k} e^{-4\pi i h \cdot S \cdot a/k} f_{k^3 H, k^3 S}(\frac{h}{k} + \frac{b}{k^2}, \frac{\tau}{k}) \tag{4.9}$$

in which the quantities a, b are defined similarly: $a = (0, \alpha^i), b = (0, \beta^i)$. We have thus reduced the averaging to the procedure in the previous section. Note that an k^2 rescaling would have been sufficient for the requirement of the averaging process because $2k^2 S(h/k + b/k^2)$ have integer entries; we used a k^3 scaling to simplify the following calculations.

We can thus calculate the average

$$\bar{f}(\frac{a}{k},\frac{b}{k},\tau)=\gamma_b+\frac{1}{k^{3N}}\sum_{(c,d)=1,d>0}d^{-2N}|\frac{\tau}{k}-\frac{c}{d}|^{-N}\sum_{h \bmod k}\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}k^3S[g+\frac{h}{k}+\frac{b}{k^2}]-4\pi ih\cdot S\cdot a/k) \tag{4.10}$$

Again, we focus on the sum of phases.

$$\begin{aligned}
&\sum_{h \bmod k}\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}k^3S[g+\frac{h}{k}+\frac{b}{k^2}]-4\pi ih\cdot S\cdot a/k)\\
=&\sum_{h \bmod k}\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}kS[kg+h+\frac{b}{k}-\frac{da}{ck^2}])\\
=&\sum_{m \bmod dk}\exp(2\pi i\frac{c}{d}kS[m+\frac{b}{k}-\frac{da}{ck^2}])\\
=&\sum_{h \bmod k}\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}kS[g+dh+\frac{b}{k}-\frac{da}{ck^2}])
\end{aligned} \tag{4.11}$$

In the second line we used the fact that $S[a]=S[b]=a\cdot S\cdot b=0$.

Thus

$$\begin{aligned}
&\sum_{h \bmod k}\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}kS[g+dh+\frac{b}{k}-\frac{da}{ck^2}])\\
=&\sum_{h \bmod k}\exp(-4\pi i\frac{d}{k}h\cdot S\cdot a)\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}kS[g+\frac{b}{k}-\frac{da}{ck^2}])
\end{aligned} \tag{4.12}$$

Hence, we can do the two sums separately.

$$\begin{aligned}
&\sum_{h \bmod k}\exp(-4\pi i\frac{d}{k}h\cdot S\cdot a)\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}kS[g+\frac{b}{k}-\frac{da}{ck^2}])\\
=&k^{2N}\delta(da \bmod k)\sum_{g \bmod d}\exp(2\pi i\frac{c}{d}kS[g+\frac{b}{k}-\frac{da}{ck^2}])\\
=&k^{2N}\delta(da \bmod k)d^N(kc,d)^N\delta(cb-\frac{da}{k} \bmod (kc,d))\\
=&k^{2N}d^N(k,d)^N\delta(cb-\frac{da}{k} \bmod (k,d))
\end{aligned} \tag{4.13}$$

in which we have omitted the exponential sum that's exactly the same as the last section. It is worth noting that in the last equality the two delta functions merge: if $da \not\equiv 0 \bmod k$ then the last delta function can never be satisfied.

We can thus plug this result back into the average:

$$\bar{f}(\frac{a}{k},\frac{b}{k},\tau) = \gamma_b + \frac{1}{k^N}\sum_{(c,d)=1,d>0} d^{-N}(k,d)^N|\frac{\tau}{k}-\frac{c}{d}|^{-N}\delta(cb-\frac{da}{k} \bmod (k,d)) \tag{4.14}$$

Similarly we write $k=(k,d)p, d=(k,d)q, (p,q)=1$. Thus

$$\begin{aligned}\bar{f}(\frac{a}{k},\frac{b}{k},\tau) &= \gamma_b + \sum_{(c,d)=1,d>0}|q\tau-pc|^{-N}\delta(pcb-qa \bmod k)\\ &= \gamma_b + \sum_{(c,d)=1,c>0}|c\tau-d|^{-N}\delta(ca-db \bmod k)\end{aligned} \tag{4.15}$$

which is the final expression for the averaged classical partition function of the twisted boson.

4.3 Shortcut via modular property

We need to observe the following properties of the quantum partition function of the twisted boson. For a modular transformation $\gamma = \begin{pmatrix} f & g \\ c & -d \end{pmatrix}$, we introduce the shorthand $\gamma\tau = \dfrac{f\tau+g}{c\tau-d}$ and . It is known in Ref. [17] that

$$|\eta(\tau)|^{-2N}\Theta_\Lambda\left(\frac{a}{k},\frac{b}{k},\tau\right) = |\eta(\gamma\tau)|^{-2N}\Theta_\Lambda\left(\frac{fa+gb}{k},\frac{ca-db}{k},\gamma\tau\right) \tag{4.16}$$

A more convenient representation is

$$\Theta_\Lambda\left(\frac{a}{k},\frac{b}{k},\tau\right) = |c\tau-d|^{-N}\Theta_\Lambda\left(\frac{fa+gb}{k},\frac{ca-db}{k},\gamma\tau\right) \tag{4.17}$$

We note that the twists transform under the modular transformation as

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} fa+gb \\ ca-db \end{pmatrix} = \gamma\cdot\begin{pmatrix} a \\ b \end{pmatrix} \tag{4.18}$$

By the Hardy-Littlewood circle method, we know that the dominant contribution of the Theta functions are given around the rational numbers of the form d/c; in particular, the

Siegel-Weil formula tells us that after averaging, the only contributions left are the singularities around rational numbers. We thus proceed to determine the singular structure of the averaged theta function, $\bar{f}$ around rational numbers.

First of all we note that

$$\bar{f}\left(\frac{a}{k}, \frac{b}{k}, \tau\right) = |c\tau - d|^{-N} \bar{f}\left(\frac{fa + gb}{k}, \frac{ca - db}{k}, \gamma\tau\right) \tag{4.19}$$

Thus

$$\bar{f}\left(\frac{a}{k}, \frac{b}{k}, \frac{d}{c} + i\epsilon\right) = |c\epsilon|^{-N} \bar{f}\left(\frac{fa + gb}{k}, \frac{ca - db}{k}, i\infty\right) \tag{4.20}$$

We note that

$$\Theta_\Lambda(r, s, \infty) = \delta(s \bmod 1) \tag{4.21}$$

And thus

$$\bar{f}(r, s, \infty) = \delta(s \bmod 1) \tag{4.22}$$

Which means that

$$\bar{f}\left(\frac{a}{k}, \frac{b}{k}, \frac{d}{c} + i\epsilon\right) = |c\epsilon|^{-N} \delta(ca - db \bmod k) \tag{4.23}$$

Define the usual notation $\Gamma = SL(2, \mathbb{Z})$. By the Siegel-Weil formula the final average is given by summing over these singularities:

$$\begin{aligned} \bar{f}\left(\frac{a}{k}, \frac{b}{k}, \tau\right) &= \frac{1}{2} \sum_{(c,d)=1} |c\tau - d|^{-N} \delta(ca - db \bmod k) \\ &= \sum_{\gamma \in \Gamma/\Gamma_0(\infty)} \tau_2^{-N/2} \mathrm{Im}(\gamma\tau)^{N/2} \delta(ca - db \bmod k) \end{aligned} \tag{4.24}$$

The $1/2$ factor accounts for the multiplicity generated by (c, d) and $(-c, -d)$. $\Gamma_0(\infty)$ is defined as

$$\Gamma_0(\infty) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, c = 0 \right\} \tag{4.25}$$

The γ_b term is obtained by taking the identity term.

4.4 Final Average: Form 1

The average of the deformed WZW partition functions is obtained by replacing the twisted boson partition functions with the averaged ones.

$$
\begin{aligned}
\bar{Z}_{WZW}(\tau) &= \frac{1}{k^N}\sum_{\alpha,\beta} Z^{PF}_{\alpha,\beta}(\tau)|\eta(\tau)|^{-2N}\bar{f}\left(\frac{a}{k},\frac{b}{k},\tau\right) \\
&= \frac{1}{k^N}\sum_{\alpha,\beta}\sum_{\gamma\in\Gamma/\Gamma_0(\infty)} Z^{PF}_{\alpha,\beta}(\tau)\frac{\tau_2^{-N/2}\mathrm{Im}(\gamma\tau)^{N/2}}{|\eta(\tau)|^{2N}}\delta(ca-db \bmod k)
\end{aligned}
\tag{4.26}
$$

in which the matrix is written as

$$
\gamma = \begin{pmatrix} f & g \\ c & -d \end{pmatrix} \tag{4.27}
$$

in which we have used the identity that

$$
\tau_2^{-N/2}\mathrm{Im}(\gamma\tau)^{N/2} = |c\tau - d|^{-N} \tag{4.28}
$$

Fix γ. Recall the action

$$
\begin{pmatrix} \nu \\ \phi \end{pmatrix} = \gamma\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} f\alpha + g\beta \\ c\alpha - d\beta \end{pmatrix} \tag{4.29}
$$

Since we know that the first N components of a and b are zero, now we also call the last N components of a and b by a and b respectively. The meaning is clear by the context; in the second context, a is the coordinates of α in the root lattice, and b is the coordinates of β in the root lattice. Since $\alpha,\beta \in Q/kQ$, $a,b \in (\mathbb{Z}/k\mathbb{Z})^N$. We note that

$$
\gamma^{-1}\gamma = I, \tag{4.30}
$$

before and after taking modulo k. Hence γ is bijection of the set of pairs (a,b) which is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^{2N}$.

On the other hand, we deduce from modular invariance of the entire WZW partition function that

$$
Z^{PF}_{\alpha,\beta}(\tau) = Z^{PF}_{\nu,\phi}(\gamma\tau) \tag{4.31}
$$

and thus

$$
\begin{aligned}
\bar{Z}_{WZW}(\tau) &= \frac{1}{k^N}\sum_{\gamma\in\Gamma/\Gamma_0(\infty)}\sum_{\nu,\phi\in Q/kQ} Z^{PF}_{\nu,\rho}(\gamma\tau)\tau_2^{-N/2}\mathrm{Im}(\gamma\tau)^{N/2}\delta(ca-db \bmod k)/|\eta(\tau)|^{2N} \\
&= \frac{1}{k^N}\sum_{\gamma\in\Gamma/\Gamma_0(\infty)}\sum_{\nu,\phi\in Q/kQ} Z^{PF}_{\nu,\phi}(\gamma\tau)\tau_2^{-N/2}\mathrm{Im}(\gamma\tau)^{N/2}\delta(\phi\in kQ)/|\eta(\tau)|^{2N} \\
&= \frac{1}{k^N}\sum_{\gamma\in\Gamma/\Gamma_0(\infty)}\sum_{\nu\in Q/kQ} \hat{Z}^{PF}_{\nu,0}(\gamma\tau)\tau_2^{-N/2}\mathrm{Im}(\gamma\tau)^{N/2}/|\eta(\tau)|^{2N} \\
&= \frac{1}{k^N}\sum_{\gamma\in\Gamma/\Gamma_0(\infty)}\sum_{\nu\in Q/kQ}\sum_{\mu\in P/kP,\lambda\in\Omega_k^+} \exp(\frac{2\pi i}{k}\mu\cdot\nu)|\eta(\gamma\tau)|^{2N}|c^\lambda_\mu(\gamma\tau)|^2\tau_2^{-N/2}\mathrm{Im}(\gamma\tau)^{N/2}/|\eta(\tau)|^{2N} \\
&= \sum_{\gamma\in\Gamma/\Gamma_0(\infty)}\sum_{\lambda\in\Omega_k^+}|c^\lambda_0(\gamma\tau)|^2\tau_2^{-N/2}\mathrm{Im}(\gamma\tau)^{N/2}\left|\frac{\eta(\gamma\tau)}{\eta(\tau)}\right|^{2D} \\
&= \sum_{\gamma\in\Gamma/\Gamma_0(\infty)}\sum_{\lambda\in\Omega_k^+}|c^\lambda_0(\gamma\tau)|^2
\end{aligned}
\tag{4.32}
$$

We note that with the particular example of $SU(N+1)_1$, $c^0_0 = \eta^{-N}$ while all other string functions vanish; furthermore we know that $SU(N+1)_1$ can be reduced to the untwisted $U(1)^N$ free boson average. Plugging the string function into the equation above reduces the answer to the result Eq. (5.8) in Ref. [2].

4.5 Final Average: Form 2 and Density of States

In this section, we fix the level k to be a prime number for simplicity.

We would like to calculate the averaged density of states of the averaged model. The importance of this procedure is that, although we are averaging theories with positive density of states, we do not want to get negative density of states due to some divergence of integrals. In Ref. [18], theories of negative density of states appear, although they have partition functions similar to Eq. (4.32).

Recall that we would like to do the following sum

$$\begin{aligned}\bar{Z}_{WZW}(\tau) &= \frac{1}{k^N}\sum_{\alpha,\beta} Z^{PF}_{\alpha,\beta}(\tau)|\eta(\tau)|^{-2N}\bar{f}\left(\frac{a}{k},\frac{b}{k},\tau\right)\\ &= \frac{|\eta(\tau)|^{-2N}}{k^N}\left(Z^{PF}_{0,0}(\tau)\bar{f}(0,0,\tau)+\sum_{\alpha\neq 0}Z^{PF}_{\alpha,0}(\tau)\bar{f}\left(\frac{a}{k},0,\tau\right)+\sum_{\alpha,\beta\neq 0}Z^{PF}_{\alpha,\beta}(\tau)\bar{f}\left(\frac{a}{k},\frac{b}{k},\tau\right)\right)\end{aligned} \tag{4.33}$$

We analyze the last term more carefully. We note that the last term where a and b are both nonzero has a delta function that contains the form $\delta(ca - bd \bmod k)$. If we view a and b as elements in $(\mathbb{Z}/k\mathbb{Z})^N$, then it can be viewed as a vector space over the field $\mathbb{Z}/k\mathbb{Z}$; we see that a and b must be colinear in the sense that $a = hb$ for some nonzero h. If not, then the term contributing must be zero.

We thus rewrite the average above as

$$\bar{Z}_{WZW}(\tau) = \frac{|\eta(\tau)|^{-2N}}{k^N}\left(Z^{PF}_{0,0}(\tau)\bar{f}(0,0,\tau)+\sum_{\alpha\neq 0}Z^{PF}_{\alpha,0}(\tau)\bar{f}\left(\frac{a}{k},0,\tau\right)+\sum_{\alpha\neq 0}\sum_{h=0}^{k-1}Z^{PF}_{h\alpha,\alpha}(\tau)\bar{f}\left(\frac{ha}{k},\frac{a}{k},\tau\right)\right) \tag{4.34}$$

We change the convention slightly from before s.t. the density of states is easier to calculate:

$$\bar{f}(0,0,\tau) = 1 + \sum_{(m,n)=1,m>0}|m\tau+n|^{-N} \tag{4.35}$$

$$\begin{aligned}\bar{f}\left(\frac{a}{k},0,\tau\right) &= 1 + \sum_{(m,n)=1,m>0}|m\tau+n|^{-N}\delta(ma = 0 \bmod k)\\ &= 1 + \sum_{(m,n)=1}|km\tau+n|^{-N}\chi_0(n)\end{aligned} \tag{4.36}$$

in which $\chi_0(n)$ is the principal character over $\mathbb{Z}/k\mathbb{Z}$: it is 0 if n can be divided by k and 1 otherwise. The second equality is the fact that m in the first line must be a multiple of k, otherwise the delta function is 0; however this means that n in the first line cannot be divided by n, otherwise the greatest common denominator is k instead of 1. And we have the third term

$$\bar{f}\left(0, \frac{a}{k}, \tau\right) = \sum_{(m,n)=1, m>0} |m\tau + n|^{-N} \delta(na = 0 \bmod k)$$
$$= \sum_{(m,n)=1} |m\tau + kn|^{-N} \chi_0(m) \tag{4.37}$$

And by the modular transformation formulas we have

$$\bar{f}\left(\frac{ha}{k}, \frac{a}{k}, \tau\right) = \bar{f}\left(0, \frac{a}{k}, \tau - h\right) = \sum_{(m,n)=1} |m(\tau - h) + kn|^{-N} \chi_0(m) \tag{4.38}$$

Now we use the trick of multiplying the series by some L function to turn this into a sum over all integers; the Dirichlet L function for a multiplicative character χ is defined as

$$L(N, \chi) = \sum_{m \in \mathbb{Z}, m>0} \frac{\chi(m)}{m^N} \tag{4.39}$$

For the particular character χ_0 above, we have the product expansion

$$L(N, \chi_0) = \frac{\prod_p (1 - p^{-N})}{1 - k^{-N}} = \frac{\zeta(N)^{-1}}{1 - k^{-N}} \tag{4.40}$$

We note that the partition functions shown in Eq. (4.36,4.37) are actually modular forms of the subgroup $\Gamma_0(k)$ defined as

$$\Gamma_0(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \middle| c = 0 \bmod k \right\} \tag{4.41}$$

which is why we are able to calculate the Fourier transforms below. We also note that the different choices of h in Eq. (4.38) combined with Eq. (4.35) is equivalent to the choice of representatives of the quotient set $\Gamma/\Gamma_0(k)$, whose cardinality is $k + 1$.

As a warmup we will do the exercise of calculating the Fourier series of $f(0, 0, \tau)$

which is explicitly stated in Ref. [7]. We multiply it by $\zeta(N)$ and get

$$
\begin{aligned}
\zeta(N)f(0,0,\tau) =&\zeta(N) + \sum_{m>0,n} |m\tau + n|^{-N} \\
=&\zeta(N) + \sum_{m>0} \sum_{c} \sum_{d=0}^{m-1} \frac{1}{m^N|\tau + c + d/m|^N} \\
=&\zeta(N) + \sum_{m>0} \sum_{d=0}^{m-1} \frac{1}{m^N}\Big(2\pi\frac{\Gamma(N-1)}{\Gamma(N/2)^2}|2\tau_2|^{1-N} \\
&+ \frac{2\pi^{N/2}}{\Gamma(N/2)} \sum_{n\neq 0} |\tau_2|^{1/2-N/2}|n|^{N/2-1/2}K_{N/2-1/2}(2\pi|n|\tau_2)\exp(2\pi in(\tau_1 + \frac{d}{m}))\Big) \\
=&\zeta(N) + 2^{2-N}\pi\tau_2^{1-N}\zeta(N-1)\frac{\Gamma(N-1)}{\Gamma(N/2)^2} \\
&+ \frac{2\pi^{N/2}}{\Gamma(N/2)} \sum_{n\neq 0} \sum_{m|n,m>0} \frac{1}{m^{N-1}}|\tau_2|^{1/2-N/2}|n|^{N/2-1/2}K_{N/2-1/2}(2\pi|n|\tau_2)\exp(2\pi in\tau_1) \\
=&\zeta(N) + 2^{2-N}\pi\tau_2^{1-N}\zeta(N-1)\frac{\Gamma(N-1)}{\Gamma(N/2)^2} \\
&+ \frac{2\pi^{N/2}}{\Gamma(N/2)} \sum_{n\neq 0} \sigma_{1-N}(n)|\tau_2|^{1/2-N/2}|n|^{N/2-1/2}K_{N/2-1/2}(2\pi|n|\tau_2)\exp(2\pi in\tau_1)
\end{aligned}
\tag{4.42}
$$

This is exactly Lemma. 5.2.11 in Ref. [7], with the application of the Legendre duplication formula. Now we denote $f_0(\tau) = \bar{f}(0,0,\tau)$.

Now we calculate the second term. We multiply it by the L function associated with character χ_0:

$$
L(N,\chi_0)\bar{f}\left(\frac{a}{k},0,\tau\right) = L(N,\chi_0) + \sum_{m,n\in\mathbb{Z},m>0} \chi_0(n)\frac{1}{|km\tau + n|^N} \tag{4.43}
$$

Now we decompose $n = kc + d$ and obtain

$$
\begin{aligned}
L(N,\chi_0)\bar{f}\left(\frac{a}{k},0,\tau\right) =& L(N,\chi_0) + \sum_{m>0}\sum_{c}\sum_{d=0}^{k-1}\chi_0(d)\frac{1}{|km\tau+kc+d|^N} \\
=& L(N,\chi_0) + \sum_{m>0}\sum_{c}\sum_{d=0}^{k-1}\frac{\chi_0(d)}{k^N}\frac{1}{|m\tau+c+\frac{d}{k}|^N} \\
=& L(N,\chi_0) + \sum_{m>0}\sum_{d=0}^{k-1}\frac{\chi_0(d)}{k^N}\Big(2\pi\frac{\Gamma(N-1)}{\Gamma(N/2)^2}|2m\tau_2|^{1-N} \\
&+ \frac{2\pi^{N/2}}{\Gamma(N/2)}\sum_{n\neq 0}|m\tau_2|^{1/2-N/2}|n|^{N/2-1/2}K_{N/2-1/2}(2\pi|n|m\tau_2)\exp(2\pi i n(m\tau_1+\frac{d}{k}))\Big) \\
=& L(N,\chi_0) + \frac{2^{2-N}\pi\Gamma(N-1)\phi(k)}{k^N\Gamma(N/2)^2}\tau_2^{1-N}\zeta(N-1) + \frac{2\pi^{N/2}}{k^N\Gamma(N/2)}\times \\
&\tau_2^{1/2-N/2}\sum_{j|k}\mu(k/j)j^{N/2+1/2}\sum_{n\neq 0}\sigma_{N-1}(n)|n|^{1/2-N/2}K_{N/2-1/2}(2\pi|n|j\tau_2)e^{2\pi i jn\tau_1} \\
=& L(N,\chi_0) + \frac{2^{2-N}\pi\Gamma(N-1)\phi(k)}{k^N\Gamma(N/2)^2}\tau_2^{1-N}\zeta(N-1) \\
&+ \frac{2\pi^{N/2}\tau_2^{1/2-N/2}}{k^N\Gamma(N/2)}\Big(-\sum_{n\neq 0}\sigma_{N-1}(n)|n|^{1/2-N/2}K_{N/2-1/2}(2\pi|n|\tau_2)e^{2\pi i n\tau_1} \\
&\quad + k^{N/2+1/2}\sum_{n\neq 0}\sigma_{N-1}(n)|n|^{1/2-N/2}K_{N/2-1/2}(2\pi|n|k\tau_2)e^{2\pi i kn\tau_1}\Big)
\end{aligned}
\tag{4.44}
$$

in which $\phi(k) = k-1$ is the Euler function, and in the fourth equality we used the identity that

$$\sum_{d=0}^{k-1}\chi_0(d)e^{2\pi i md/k} = \sum_{r|(m,k)}\mu\left(\frac{k}{r}\right)r \tag{4.45}$$

The last step comes from definition of the Mobius function. The derivation above reproduces the derivation in Ref. [15]. We note that the final fourier transform above is independent of the choice of the temporal twist a. We thus define $f(\tau) = \bar{f}\left(\frac{a}{k},0,\tau\right)$.

For the other term, we also use the trick to get first

$$L(N,\chi_0)\bar{f}\left(0,\frac{a}{k},\tau\right) = \sum_{m,n\in\mathbb{Z},m>0}\chi_0(m)\frac{1}{|m\tau+kn|^N} = \sum_{(m,n)\in Z^2,m>0}\frac{\chi_0(m)}{k^N}\frac{1}{|m\tau/k+n|^N} \tag{4.46}$$

For this term, we decompose $n = mc + d$, and get

$$\begin{aligned}
L(N,\chi_0)\bar{f}\left(0,\frac{a}{k},\tau\right) &= \sum_{m>0}\sum_{c}\sum_{d=0}^{m-1}\frac{\chi_0(m)}{k^N}\frac{1}{|m\tau/k+mc+d|^N}\\
&= \sum_{m>0}\sum_{c}\sum_{d=0}^{m-1}\frac{1}{k^N}\frac{\chi_0(m)}{m^N}\frac{1}{|\frac{\tau}{k}+c+\frac{d}{m}|^N}\\
&= \sum_{m>0}\sum_{d=0}^{m-1}\frac{1}{k^N}\frac{\chi_0(m)}{m^N}\left(2\pi\frac{\Gamma(N-1)}{\Gamma(N/2)^2}\left|\frac{2\tau_2}{k}\right|^{1-N}\right.\\
&\quad\left.+\frac{2\pi^{N/2}}{\Gamma(N/2)}\sum_{n\neq 0}\left|\frac{\tau_2}{k}\right|^{1/2-N/2}|n|^{N/2-1/2}K_{N/2-1/2}\left(2\pi\frac{|n|\tau_2}{k}\right)\exp\left(2\pi i n\left(\frac{\tau_1}{k}+\frac{d}{m}\right)\right)\right)\\
&=\frac{1}{k^N}\frac{2\pi^{N/2}}{\Gamma(N/2)}\sum_{n\neq 0}\sum_{m|n,m>0}\frac{\chi_0(m)}{m^{N-1}}\left|\frac{\tau_2}{k}\right|^{1/2-N/2}|n|^{N/2-1/2}K_{N/2-1/2}\left(2\pi\frac{|n|\tau_2}{k}\right)\exp\left(2\pi i n\frac{\tau_1}{k}\right)\\
&\quad+\sum_{m>0}\frac{2\pi m}{k^N}\frac{\Gamma(N-1)}{\Gamma(N/2)^2}\left|\frac{2\tau_2}{k}\right|^{1-N}\\
&=\frac{2\pi^{N/2}|\tau_2|^{1/2-N/2}}{k^{N/2+1/2}\Gamma(N/2)}\sum_{n\neq 0}|n|^{N/2-1/2}\sigma_{1-N}(n,\chi_0)K_{N/2-1/2}\left(2\pi\frac{|n|\tau_2}{k}\right)\exp\left(2\pi i n\frac{\tau_1}{k}\right)\\
&\quad+\frac{2^{2-N}\pi\Gamma(N-1)\tau_2^{1-N}}{k\Gamma(N/2)^2}L(N-1,\chi_0)
\end{aligned}\tag{4.47}$$

where

$$\sigma_{1-D}(n,\chi_0) = \sum_{m|n,m>0} m^{1-D}\chi_0(m) \tag{4.48}$$

is the divisor function. Again, we note that the final result is independent of a. We define $g(\tau) = \bar{f}\left(0,\frac{a}{k},\tau\right)$. We note that $g(\tau-h) = \bar{f}\left(\frac{ha}{k},\frac{a}{k},\tau\right)$.

Furthermore, we note that f_0 and f have period 1, while the function g has period k. This will be important in the discussion later.

We would like to do the following sum

$$\bar{Z}_{WZW}(\tau) = \frac{1}{k^N}\sum_{\alpha,\beta}Z^{PF}_{\alpha,\beta}(\tau)|\eta(\tau)|^{-2N}\bar{f}\left(\frac{a}{k},\frac{b}{k},\tau\right) = \frac{1}{k^N}\sum_{\lambda,\mu}\sum_{\alpha,\beta}e^{\pi i\alpha\cdot(2\mu-\beta-B_0\beta)/k}c^{\lambda}_{\mu}\bar{c}^{\lambda}_{\mu+\beta}\bar{f}\left(\frac{a}{k},\frac{b}{k},\tau\right) \tag{4.49}$$

The formula for c^{λ}_{μ} is generally hard. Again, we would like to break the terms above into the three parts as Eq. (4.34). The first term is trivial: it is simply $\sum_{\lambda,\mu} |c^\lambda_\mu(\tau)|^2 f_0(\tau)/k^N$.

As for the second term, we note that

$$\frac{1}{k^N}\sum_{\lambda,\mu}\sum_{\alpha\neq 0} e^{2\pi i\alpha\cdot\mu/k}|c^\lambda_\mu|^2 f(\tau) = \frac{f(\tau)}{k^N}\left(k^N\sum_\lambda |c^\lambda_0|^2 - \sum_{\lambda,\mu}|c^\lambda_\mu|^2\right) \tag{4.50}$$

Suppose the fourier transform of $g(\tau)$ is

$$g(\tau) = \sum_{l=-\infty}^{\infty} g_l(\tau_2)e^{2\pi i l\frac{\tau_1}{k}} \tag{4.51}$$

in which we can read out that if $l = 0$,

$$g_0(\tau_2) = \frac{2^{2-N}\pi\Gamma(N-1)\tau_2^{1-N}}{k\Gamma(N/2)^2}\frac{L(N-1,\chi_0)}{L(N,\chi_0)} \tag{4.52}$$

and

$$g_l(\tau_2) = \frac{2\pi^{N/2}\,|\tau_2|^{1/2-N/2}}{k^{N/2+1/2}\Gamma(N/2)L(N,\chi_0)}|l|^{N/2-1/2}\sigma_{1-N}(l,\chi_0)K_{N/2-1/2}\left(2\pi\frac{|l|\tau_2}{k}\right) \tag{4.53}$$

And thus

$$g(\tau-h) = \sum_{l=-\infty}^{\infty} g_l(\tau_2)e^{2\pi i l\frac{\tau_1-h}{k}} \tag{4.54}$$

which means that

$$\begin{aligned}&\frac{|\eta(\tau)|^{-2N}}{k^N}\sum_{\alpha\neq 0}\sum_{h=0}^{k-1} Z^{PF}_{h\alpha,\alpha}(\tau)\bar{f}\left(\frac{ha}{k},\frac{a}{k},\tau\right)\\ =&\frac{1}{k^N}\sum_{\alpha\neq 0}\sum_{h=0}^{k-1}\sum_{\lambda,\mu} e^{-\pi i h\alpha\cdot(2\mu+\alpha)/k}c^\lambda_\mu\bar{c}^\lambda_{\mu+\alpha}\sum_{l=-\infty}^{\infty} g_l(\tau_2)e^{2\pi i l\frac{x-h}{k}}\\ =&k^{1-N}\sum_{\alpha\neq 0}\sum_{\lambda,\mu}\sum_{l=-\infty}^{\infty} c^\lambda_\mu\bar{c}^\lambda_{\mu+\alpha}g_l(\tau_2)e^{2\pi i l\frac{\tau_1}{k}}\delta(l+\alpha\cdot\mu+|\alpha|^2/2 \bmod k)\end{aligned} \tag{4.55}$$

Collecting all the terms, we have

$$\begin{aligned}\bar{Z}_{WZW}(\tau) =& \sum_\lambda |c^\lambda_0|^2\left(\frac{f_0(\tau)}{k^N} + \left(1-\frac{1}{k^N}\right)f(\tau)\right) + \sum_{\lambda,\mu\neq 0}|c^\lambda_\mu|^2\frac{f_0(\tau)-f(\tau)}{k^N}\\ &+ k^{1-N}\sum_{\alpha\neq 0}\sum_{\lambda,\mu}\sum_{l=-\infty}^{\infty} c^\lambda_\mu\bar{c}^\lambda_{\mu+\alpha}g_l(\tau_2)e^{2\pi i l\frac{\tau_1}{k}}\delta(l+\alpha\cdot\mu+|\alpha|^2/2 \bmod k)\end{aligned} \tag{4.56}$$

We thus define

$$h_1(\tau) = \frac{f_0(\tau)}{k^N} + \left(1 - \frac{1}{k^N}\right) f(\tau), h_2(\tau) = \frac{f_0(\tau) - f(\tau)}{k^N} \tag{4.57}$$

We then study the properties of h_1 and h_2.

We note that

$$\begin{aligned} f(\tau) =& 1 + \frac{2^{2-N}\pi\Gamma(N-1)\phi(k)\tau_2^{1-N}}{(k^N-1)\Gamma(N/2)^2}\frac{\zeta(N-1)}{\zeta(N)} \\ &+ \frac{2\pi^{N/2}\tau_2^{1/2-N/2}}{(k^N-1)\Gamma(N/2)\zeta(N)}\Big(-\sum_{n\neq 0}\sigma_{1-N}(n)|n|^{N/2-1/2}K_{N/2-1/2}(2\pi|n|\tau_2)e^{2\pi i n\tau_1} \\ &+ k^{N/2+1/2}\sum_{n\neq 0}\sigma_{1-N}(n)|n|^{N/2-1/2}K_{N/2-1/2}(2\pi|n|k\tau_2)e^{2\pi i k n\tau_1}\Big) \end{aligned} \tag{4.58}$$

in which we used the property that

$$\sigma_{1-N}(n)|n|^{N/2-1/2} = \sigma_{N-1}(n)|n|^{1/2-N/2} \tag{4.59}$$

and the relationship between $\zeta(N)$ and $L(D,\chi_0)$.

And thus

$$\begin{aligned} h_1(\tau) =& \frac{f_0(\tau)}{k^N} + \left(1 - \frac{1}{k^N}\right) f(\tau) \\ =& 1 + \frac{\pi 2^{2-N}\tau_2^{1-N}}{k^N}\frac{\zeta(N-1)}{\zeta(N)}\frac{\Gamma(N-1)}{\Gamma(N/2)^2} + \phi(k)\frac{2^{2-N}\pi\tau_2^{1-N}}{k^N}\frac{\zeta(N-1)}{\zeta(N)}\frac{\Gamma(N-1)}{\Gamma(N/2)^2} \\ &+ \frac{2\pi^{N/2}\tau_2^{1/2-N/2}}{k^N\Gamma(N/2)\zeta(N)}k^{N/2+1/2}\sum_{n\neq 0}\sigma_{1-N}(n)|n|^{N/2-1/2}K_{N/2-1/2}(2\pi|n|k\tau_2)e^{2\pi i k n\tau_1} \\ =& 1 + \frac{2^{2-N}\pi\tau_2^{1-N}}{k^{N-1}}\frac{\zeta(N-1)}{\zeta(N)}\frac{\Gamma(N-1)}{\Gamma(N/2)^2} \\ &+ \frac{2\pi^{N/2}\tau_2^{1/2-N/2}}{k^N\Gamma(N/2)\zeta(N)}k^{N/2+1/2}\sum_{n\neq 0}\sigma_{1-N}(n)|n|^{N/2-1/2}K_{N/2-1/2}(2\pi|n|k\tau_2)e^{2\pi i k n\tau_1} \end{aligned} \tag{4.60}$$

For $h_2(\tau)$, we have

$$\begin{aligned}h_2(\tau) =&\frac{f_0(\tau)-f(\tau)}{k^N}\\ =&\frac{2^{2-N}\pi\Gamma(N-1)\tau_2^{1-N}}{k^N\Gamma(N/2)^2}\frac{\zeta(N-1)}{\zeta(N)}\left(1-\frac{\phi(k)}{k^N-1}\right)\\ &+\frac{2\pi^{N/2}\tau_2^{1/2-N/2}}{(k^N-1)\Gamma(N/2)\zeta(N)}\left(\sum_{n\neq 0,k\nmid n}\sigma_{1-N}(n)|n|^{N/2-1/2}K_{N/2-1/2}(2\pi|n|k\tau_2)e^{2\pi ikn\tau_1}\right.\\ &\left.+\sum_{n\neq 0}\left(\sigma_{N-1}(kn)-\sigma_{N-1}(n)\right)|kn|^{1/2-N/2}K_{N/2-1/2}(2\pi|n|k\tau_2)e^{2\pi ikn\tau_1}\right)\end{aligned} \tag{4.61}$$

And thus in the end we have

$$\bar{Z}_{WZW}(\tau)=\sum_{\lambda}|c_0^\lambda|^2h_1(\tau)+\sum_{\lambda,\mu\neq 0}|c_\mu^\lambda|^2h_2(\tau)+k^{1-N}\sum_{\lambda,\mu,\alpha\neq 0}\sum_{l=-\infty}^{\infty}c_\mu^\lambda\overline{c}_{\mu+\alpha}^\lambda g_l(\tau_2)e^{2\pi il\frac{\tau_1}{k}}\delta(l+\alpha\cdot\mu+|\alpha|^2/2 \bmod k) \tag{4.62}$$

We note that in the last term of the expression above, although the $\exp(2\pi il\tau_1/k)$ might not be periodic when $\tau_1\to\tau_1+1$,

$$c_\mu^\lambda\overline{c}_{\mu+\alpha}^\lambda = q^{m_\lambda(\hat{\mu})}\bar{q}^{m_\lambda(\hat{\mu}+\hat{\alpha})}\sum_{n=0}^{\infty}a_nq^n = e^{2\pi i\tau_1(|\lambda+\alpha|^2-|\lambda|^2)/2k}e^{-2\pi\tau_2(m_\lambda(\hat{\mu})+m_\lambda(\hat{\mu}+\hat{\alpha}))}\sum_{n=0}^{\infty}a_nq^n \tag{4.63}$$

Combining the two equations above, we find that the last term is explicitly periodic when $\tau_1\to\tau_1+1$. This concludes our calculation of the averaged partition function.

We note that since the functions K have positive Laplace transform, and all coefficients in h_1, h_2 are positive, the density of states is given by the convolution of the density of states with the Laplace transforms of K. Since the convolution of two positive functions is positive, we conclude that our averaged theory has positive density of states.

CHAPTER 5
CONCLUSION

In conclusion, we have computed the average partition function of the deformed $SU(N+1)_k$ WZW model over the Narain moduli space. As it bears the form of a Poincaré series, it strongly suggests the possibility to interpret it as a gravitational theory, in particular, a $U(1)$ Chern-Simons gravity theory mentioned in Ref. [2] coupled to matter fields similar to parafermions.

Using the ideas presented in this thesis, one can in general carry out averaging procedures completely similar in other scenarios. For example, current-current deformations in supersymmetric CFTs are also marginal; one can average over those deformations by writing the theories as an orbifold theory of the bosons and another theory, and then average over the bosons. This would generate a large family of possible dualities between gravitational theories and an ensemble average of CFTs.

Another important aspect is to compute physical quantities, such as correlation functions, in the averaged theory. We note that fields corresponding to the bosons in such theories are moduli dependent and thus their correlation functions are hard to compute after the average; luckily, in this scenario, the parafermion fields are independent of the moduli. Another possibility is to consider the twist field; a calculation of correlation functions of $\mathbb{Z}_2$ twist fields is recently studied in Ref. [5]. We should also pay attention to the correlation functions of partition functions $\langle ZZ \rangle$ as well, which may hint at the existence of wormholes in the corresponding gravitational theories. A further attempt is to study averaged theories with chemical potentials; the most recent progress in this direction made in the bosonic theories is Ref. [9]. It would also be fruitful to calculate the density of states for non prime levels and higher genus partition function; in particular, if a holographic bulk gravity theory were to be constructed, we would have to know about higher genus partition functions on the boundary.

APPENDIX A

HOW TO AVOID OVERCOUNTING THE VACUUM

We recall that in Ref. [12, 13] the twisted partition function of the parafermions is defined as

$$Z^{PF'}_{\alpha,\beta} = |\eta(q)|^{2N} \sum_{\lambda\in\Omega_k^+} \sum_{\mu\in P/kQ} e^{-\pi i\alpha\cdot(2\mu-\beta)/k} c^\lambda_\mu(q)\bar{c}^\lambda_{\mu-\beta}(\bar{q}) \tag{A.1}$$

We find that the phase is ambiguous: it is representative dependent. That problem is solved by including the B_0 field, as shown in Section 3.1.2. We further find that the coefficient of the vacuum term of this partition function does not have modulus 1, which is unnatural. Our definition for the parafermion partition function, which avoids both of these caveats, is

$$Z^{PF}_{\alpha,\beta} = |\eta(q)|^{2N} \sum_{\lambda\in\Omega_k^+} \sum_{\mu\in P/kP} e^{-\pi i\alpha\cdot(2\mu-\beta)/k+\pi i\alpha^i(B_1)_{ij}\beta^j/k} c^\lambda_\mu(q)\bar{c}^\lambda_{\mu-\beta}(\bar{q}) \tag{A.2}$$

so we aim to prove that for fixed representatives α, β,

$$(N+1)Z^{PF}_{\alpha,\beta} = e^{\pi i\alpha^i(B_1)_{ij}\beta^j/k} Z^{PF'}_{\alpha,\beta} \tag{A.3}$$

The derivation makes use of the outer automorphisms. Recall that with $\hat{su}(N+1)_k$ the automorphism group $O(\hat{su}(N+1)_k) \cong \mathbb{Z}_{N+1}$; denote the generator by a. a^n applies an the action of the outer automorphisms apply a Weyl transformation w_{a^n} to the weights and then shifts it by $\delta_n \equiv k\omega_{a^n}$, with the temporary definition that $\omega_0 = 0$.

And thus, since $\{k\omega_i\} = kP/kQ$, we have for fixed $\mu \in P/kP$ and $\delta_A \in kP/kQ$:

$$\begin{aligned}
&\sum_{\lambda\in\Omega_k^+} c^\lambda_{\mu+\delta_A}\bar{c}^\lambda_{\mu+\delta_A-\beta} \\
&= \sum_{\lambda\in\Omega_k^+} c^{A^{-1}\lambda}_{w_A^{-1}\mu}\bar{c}^{A^{-1}\lambda}_{w_A^{-1}(\mu-\beta)} \\
&= \sum_{\lambda\in\Omega_k^+} c^{A^{-1}\lambda}_{\mu}\bar{c}^{A^{-1}\lambda}_{\mu-\beta} \\
&= \sum_{\lambda\in\Omega_k^+} c^{\lambda}_{\mu}\bar{c}^{\lambda}_{\mu-\beta}
\end{aligned} \tag{A.4}$$

In the second and third line we used Eq. (2.51); in the last line we used the fact that A is a symmetry of the set Ω_k^+.

Now we see why the vacuum is overcounted. Basically, originally the vacuum term is given by $|c_0^0|^2$. However, by applying outer automorphisms, $|c_{k\omega_i}^{k\omega_i}|^2 = |c_0^0|^2$ also contributes the same to the vacuum coefficient.

And thus

$$\begin{aligned}
(N+1)Z_{\alpha,\beta}^{PF} &= (N+1)|\eta(q)|^{2N} \sum_{\lambda\in\Omega_k^+} \sum_{\mu\in P/kP} e^{-\pi i\alpha\cdot(2\mu-\beta)/k+\pi i\alpha^i(B_1)_{ij}\beta^j/k} c_\mu^\lambda(q)\bar{c}_{\mu-\beta}^\lambda(\bar{q}) \\
&= |\eta(q)|^{2N} \sum_{\lambda\in\Omega_k^+} \sum_{\delta\in kP/kQ} \sum_{\mu\in P/kP} e^{-\pi i\alpha\cdot(2\mu+2\delta-\beta)/k+\pi i\alpha^i(B_1)_{ij}\beta^j/k} c_{\mu+\delta}^\lambda(q)\bar{c}_{\mu+\delta-\beta}^\lambda(\bar{q}) \\
&= |\eta(q)|^{2N} \sum_{\lambda\in\Omega_k^+} \sum_{\mu\in P/kQ} e^{-\pi i\alpha\cdot(2\mu-\beta)/k+\pi i\alpha^i(B_1)_{ij}\beta^j/k} c_{\mu+\delta}^\lambda(q)\bar{c}_{\mu+\delta-\beta}^\lambda(\bar{q}) \\
&= e^{\pi i\alpha^i(B_1)_{ij}\beta^j/k} Z_{\alpha,\beta}^{PF\prime}
\end{aligned} \tag{A.5}$$

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